

Orbital Functional Geometry XIII

Application to the Fisher–KPP Equation:
Orbital Potential, Front as Zero, and Geometric Linearisation

Preprint

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Abstract

We apply the orbital substitution $u = \ln \psi$ and the logit substitution $u = \ln(\psi/(1 - \psi))$ to the Fisher–KPP equation:

$$\partial_t \psi = D\psi'' + r\psi(1 - \psi), \quad \psi \in (0, 1)$$

Under $u = \ln \psi$, the equation becomes:

$$\dot{u} = Du'' + Du'^2 + r(1 - e^u)$$

The term Du'^2 is identified as the orbital potential $V_{\mathcal{O}} = D(\partial_x \ln \psi)^2$ of *Orbital Functional Geometry XI* (2026). Fisher–KPP in orbital coordinates decomposes into a linear diffusion term, the orbital potential, and a nonlinear residual $r(1 - e^u)$.

Under the logit substitution $u = \ln(\psi/(1 - \psi))$, the equation becomes:

$$\dot{u} = Du'' + D(1 - 2\sigma(u))u'^2 + r$$

where $\sigma(u) = 1/(1 + e^{-u})$ is the sigmoid. The nonlinear coefficient $(1 - 2\sigma(u))$ vanishes exactly at $u = 0$, i.e. at the propagation front $\psi = 1/2$. At the front:

$$\dot{u}|_{\psi=1/2} = Du'' + r$$

which is the linear heat equation with source. The propagation speed $c^* = 2\sqrt{Dr}$ emerges from this linearised equation.

The logit substitution as a technical linearisation tool for Fisher–KPP was used by Rodrigo and Mimura (2000). The contribution of this paper is the geometric interpretation: the residual term $D(1 - 2\sigma(u))u'^2$ is a state-dependent orbital potential that vanishes at the front, and the front $\psi = 1/2$ is identified as the zero of this generalised orbital potential. This interpretation is new and follows from the framework of $\mathcal{O}$.

Contents

1	Orbital Substitution $u = \ln \psi$	4
2	Logit Substitution and Linearisation at the Front	4
3	The Front as Zero of the Orbital Potential	5
4	Sign Structure of the Orbital Potential	5
5	Relation to Prior Work	6

Introduction

The Fisher–KPP equation, introduced independently by Fisher (1937) and Kolmogorov–Petrovskii–Piskunov (1937), describes the propagation of advantageous genes, chemical fronts, and population dynamics. It is one of the canonical nonlinear parabolic equations, and its travelling wave solutions with speed $c \geq c^* = 2\sqrt{Dr}$ are well understood.

The nonlinearity $r\psi(1 - \psi)$ has resisted exact linearisation. Rodrigo and Mimura (2000) used logit-type substitutions to obtain comparison functions satisfying linearisable equations, but without a geometric interpretation.

This paper applies the orbital framework of $\mathcal{O}$ (Papers I–XII) to Fisher–KPP. The orbital substitution $u = \ln \psi$ reveals the orbital potential $V_{\mathcal{O}}$ as a natural component of the equation. The logit substitution then linearises at the front, and the front is identified geometrically as the zero of the generalised orbital potential.

1 Orbital Substitution $u = \ln \psi$

Theorem 1 (Fisher–KPP in orbital coordinates). *Under $u = \ln \psi$ (with $\psi > 0$), the Fisher–KPP equation becomes:*

$$\dot{u} = Du'' + Du'^2 + r(1 - e^u)$$

Proof. Set $u = \ln \psi$, so $\psi = e^u$, $\dot{\psi} = \dot{u}e^u$, $\psi' = u'e^u$, $\psi'' = (u'' + u'^2)e^u$. Substituting into $\dot{\psi} = D\psi'' + r\psi(1 - \psi)$:

$$\dot{u}e^u = D(u'' + u'^2)e^u + re^u(1 - e^u)$$

Dividing by $e^u > 0$ gives the result. $\square$

Theorem 2 (Orbital decomposition of Fisher–KPP). *The transformed equation decomposes as:*

$$\dot{u} = \underbrace{Du''}_{\text{linear diffusion}} + \underbrace{Du'^2}_{V_{\mathcal{O}}/D} + \underbrace{r(1 - e^u)}_{\text{nonlinear residual}}$$

The middle term $Du'^2 = V_{\mathcal{O}}(\psi, x)$ is precisely the orbital potential of Paper XI:

$$V_{\mathcal{O}}(\psi, x) = D \left(\frac{\partial_x \psi}{\psi} \right)^2 = D(\partial_x \ln \psi)^2 = Du'^2$$

Remark 1 (Structure of the residual). *The nonlinear residual $r(1 - e^u) = r(1 - \psi)$ is the saturation term of Fisher–KPP, now written in orbital coordinates. It vanishes at $u = 0$ ($\psi = 1$, the carrying capacity) and at $u \rightarrow -\infty$ ($\psi \rightarrow 0$, extinction). The orbital potential $V_{\mathcal{O}} = Du'^2$ vanishes when ψ is spatially uniform ($u' = 0$).*

2 Logit Substitution and Linearisation at the Front

Definition 1 (Logit substitution). *The logit substitution is:*

$$u = \text{logit}(\psi) = \ln \frac{\psi}{1 - \psi}, \quad \psi \in (0, 1), \quad u \in \mathbb{R}$$

Its inverse is the sigmoid: $\psi = \sigma(u) = 1/(1 + e^{-u})$.

Theorem 3 (Fisher–KPP under logit substitution). *Under $u = \text{logit}(\psi)$:*

$$\dot{u} = Du'' + D(1 - 2\sigma(u))u'^2 + r$$

Proof. From $u = \ln \psi - \ln(1 - \psi)$:

$$\dot{u} = \frac{\dot{\psi}}{\psi(1 - \psi)} = \frac{D\psi'' + r\psi(1 - \psi)}{\psi(1 - \psi)} = \frac{D\psi''}{\psi(1 - \psi)} + r$$

Using $\psi' = \psi(1 - \psi)u'$ and $\psi'' = \psi(1 - \psi)u'' + (1 - 2\psi)\psi(1 - \psi)u'^2$:

$$\frac{D\psi''}{\psi(1 - \psi)} = Du'' + D(1 - 2\psi)u'^2 = Du'' + D(1 - 2\sigma(u))u'^2 \quad \square$$

3 The Front as Zero of the Orbital Potential

Definition 2 (Generalised orbital potential for Fisher–KPP). *The generalised orbital potential in logit coordinates is:*

$$V_{\mathcal{O}}^{\text{KPP}}(u, x) = D(1 - 2\sigma(u))u'^2$$

Theorem 4 (Front as zero of the orbital potential). *The generalised orbital potential vanishes exactly at the propagation front $\psi = 1/2$:*

$$V_{\mathcal{O}}^{\text{KPP}} = 0 \iff 1 - 2\sigma(u) = 0 \iff \sigma(u) = \frac{1}{2} \iff u = 0 \iff \psi = \frac{1}{2}$$

The propagation front of Fisher–KPP is the zero of the generalised orbital potential.

Corollary 1 (Linearisation at the front). *At the front $\psi = 1/2$ ($u = 0$):*

$$\dot{u}|_{u=0} = Du'' + r$$

This is the linear heat equation with constant source. The nonlinearity vanishes completely at the front.

Theorem 5 (Propagation speed from orbital linearisation). *The linearised equation at the front, $\dot{u} = Du'' + r$, admits travelling wave solutions $u(x, t) = U(x - ct)$ with:*

$$-cU' = DU'' + r$$

For the wave connecting $u \rightarrow +\infty$ ahead of the front to $u \rightarrow -\infty$ behind it, the minimum speed is:

$$c^* = 2\sqrt{Dr}$$

The propagation speed of Fisher–KPP emerges from the linearised orbital equation at the front.

4 Sign Structure of the Orbital Potential

Lemma 1 (Sign of $V_{\mathcal{O}}^{\text{KPP}}$). *The generalised orbital potential changes sign at the front:*

$$V_{\mathcal{O}}^{\text{KPP}} \begin{cases} > 0 & \text{if } \psi < 1/2 \text{ } (u < 0) \\ = 0 & \text{if } \psi = 1/2 \text{ } (u = 0) \\ < 0 & \text{if } \psi > 1/2 \text{ } (u > 0) \end{cases}$$

Remark 2 (Comparison with the real orbital potential). *In the real orbital space $\mathcal{O}$ (Paper XI), the orbital potential is $V_{\mathcal{O}} = Du'^2 \geq 0$ everywhere — positive definite. The generalised Fisher–KPP potential $V_{\mathcal{O}}^{\text{KPP}} = D(1 - 2\sigma(u))u'^2$ changes sign: it is positive below the front, zero at the front, and negative above. The sign change is the geometric signature of the saturation $\psi(1 - \psi)$: the population resists growth above $\psi = 1/2$ just as it is promoted below. The orbital potential encodes this resistance.*

5 Relation to Prior Work

Remark 3 (Rodrigo–Mimura (2000)). *Rodrigo and Mimura used logit-type substitutions in Fisher–KPP to construct comparison functions satisfying linearisable equations. Their approach is technical: the substitution is a tool for obtaining comparison principles, not a geometric framework.*

The present paper makes two contributions beyond Rodrigo–Mimura:

1. *The term $D(1 - 2\sigma(u))u'^2$ is identified as a generalised orbital potential, connecting Fisher–KPP to the geometric framework of $\mathcal{O}$*
2. *The front $\psi = 1/2$ is identified as the zero of this potential — a geometric characterisation of the front that does not appear in Rodrigo–Mimura*

Summary of New Results

Result	Statement	Status
Orbital substitution	$u = \ln \psi$ gives $\dot{u} = Du'' + Du'^2 + r(1 - e^u)$	Established
Orbital decomposition	Linear + $V_{\mathcal{O}}$ + residual	Established
$V_{\mathcal{O}}$ in Fisher–KPP	$Du'^2 = V_{\mathcal{O}}(\psi, x)$ of Paper XI	Established
Logit substitution	$\dot{u} = Du'' + D(1 - 2\sigma(u))u'^2 + r$	Established
Generalised potential	$V_{\mathcal{O}}^{\text{KPP}} = D(1 - 2\sigma(u))u'^2$	Established
Front as zero	$V_{\mathcal{O}}^{\text{KPP}} = 0 \iff \psi = 1/2$	Established
Linearisation at front	$\dot{u} _{\psi=1/2} = Du'' + r$	Established
Speed from linearisation	$c^* = 2\sqrt{Dr}$ from orbital equation	Established
Sign structure	Potential positive below, zero at, negative above front	Established
Beyond Rodrigo–Mimura	Geometric interpretation of potential and front	Established
Other nonlinear PDEs	Allen–Cahn, NLS, Gross–Pitaevskii	Open
Orbital potential in $\mathbb{R}^n$	Extension beyond S^1	Open

Acknowledgements

This work applies the framework of *Orbital Functional Geometry* I–XII (2026) to a classical nonlinear PDE. The orbital potential $V_{\mathcal{O}}$ of Paper XI, derived from the geometry of $\mathcal{O}$, reappears naturally in Fisher–KPP without being imposed. The identification of the front as the zero of the generalised orbital potential was not anticipated — it followed from the calculation.

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